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Configurations and parallelograms associated to centers of mass

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The purpose of this article is to

(1) define $M(t,k)$ the t–fold center of mass arrangement for k points in the plane,
(2) give elementary properties of $M(t,k)$ and
(3) give consequences concerning the space $M(2,k)$ of k distinct points in the plane, no four of which are the vertices of a parallelogram.

The main result proven in this article is that the classical unordered configuration of k points in the plane is not a retract up to homotopy of the space of k unordered distinct points in the plane, no four of which are the vertices of a parallelogram. The proof below is homotopy theoretic without an explicit computation of the homology of these spaces.

In addition, a second, speculative part of this article arises from the failure of these methods in the case of odd primes p. This failure gives rise to a candidate for the localization at odd primes p of the double loop space of an odd sphere obtained from the p–fold center of mass arrangement. Potential consequences are listed.

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1 Introduction and statement of results

Fix integers k and t. The t–fold center of mass arrangement $M(t,k)$ for integers t with $k \geq t \geq 1$ is defined as the subspace of the k–fold product $\mathbb{C}^k$ given by ordered k–tuples of points $(x_1,\dots,x_k)$ such that the centroids of any set of t elements in the underlying set $\{x_1,\dots,x_k\}$

$$\sigma_t(x_{i_1},x_{i_2},\dots,x_{i_t}) = (1/t)(x_{i_1}+x_{i_2}+\cdots+x_{i_t})$$

are distinct for all distinct subsets $\{x_{i_1},x_{i_2},\dots,x_{i_t}\}$, and $\{x_{j_1},x_{j_2},\dots,x_{j_t}\}$. In particular, $M(t,k)$ is the complement of the union of the hyperplanes specified by

$$\sigma_t(x_{i_1},x_{i_2},\dots,x_{i_t}) - \sigma_t(x_{j_1},x_{j_2},\dots,x_{j_t}) = 0$$

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for all pairs of unequal sets $S_I = \{x_{i_1}, x_{i_2}, \dots, x_{i_t}\}$, and $S_J = \{x_{j_1}, x_{j_2}, \dots, x_{j_t}\}$. Write

$$|S_J|$$

for the cardinality of the set S_J. In case $k < t$, define $M(t,k)$ to be the Fadell–Neuwirth configuration space $\mathrm{Conf}(\mathbb{C}, k)$ of ordered k tuples of distinct points in $\mathbb{C}$ (see Fadell–Neuwirth [6]).

Finite unions of complex hyperplanes in complex k–space are known as complex hyperplane arrangements in Orlik–Terao [8]. The space $M(t,k)$ is a complement of a complex hyperplane arrangement. Consider an equivalent formulation of $M(t,k)$ as the complement of the variety $V(t,k)$ of ordered k–tuples $(x_1, \dots, x_k)$ defined by the equation

$$\prod_{S_I \neq S_J, |S_I| = |S_J| = t} ([x_{i_1} + x_{i_2} + \cdots + x_{i_t}] - [x_{j_1} + x_{j_2} + \cdots + x_{j_t}]) = 0$$

with

$$M(t,k) = \mathbb{C}^k - V(t,k).$$

Modifications of the $M(t,k)$, $M'(t,k)$, are defined as follows:

$$M'(t,k) = \cap_{1 \leq s \leq t} M(s,k).$$

Thus $M'(t,k)$ is the complement of the variety $W(t,k)$ of ordered k–tuples $(x_1, \dots, x_k)$ defined by the equation

$$\prod_{S_I \neq S_J, 1 < q = |S_I| = |S_J| \leq t} \big([x_{i_1} + x_{i_2} + \cdots + x_{i_q}] - [x_{j_1} + x_{j_2} + \cdots + x_{j_q}]\big) = 0$$

with

$$M'(t,k) = \mathbb{C}^k - W(t,k).$$

Similarly, if $k < t$, define $M'(t,k)$ to be $\mathrm{Conf}(\mathbb{C}, k)$.

In addition, there are natural inclusions

$$M'(t,k) \longrightarrow M(t,k) \longrightarrow \mathrm{Conf}(\mathbb{C}, k).$$

These inclusions are equivariant with respect to the natural action of the symmetric group on k letters, Σ_k.

Consider the t–fold symmetric product $\mathbb{C}^t/\Sigma_t$, and notice that there is a map

$$\chi_t \colon \ \mathrm{Conf}(\mathbb{C}, k) \to (\mathbb{C}^t/\Sigma_t)^{\binom{k}{t}}$$

gotten by choosing all t–element subsets out of a set of cardinality k with a fixed ordering of the subsets. The map χ_t is given on the level of points by the formula

$$\chi_t(z_1, z_2, \dots, z_k) = \prod_{i_1<i_2<\ldots<i_t} [z_{i_1}, z_{i_2}, \dots, z_{i_t}]$$

for which the points $[z_{i_1}, z_{i_2}, \dots, z_{i_t}]$ in $\mathbb{C}^t/\Sigma_t$ are ordered in the product left lexicographically by indices and over all subsets of cardinality t in the set $\{z_1, z_2, \dots, z_k\}$.

Notice that the map $\chi_t\colon\ \mathrm{Conf}(\mathbb{C}, k) \to (\mathbb{C}^t/\Sigma_t)^{\binom{k}{t}}$ takes values in the configuration space $\mathrm{Conf}(\mathbb{C}^t/\Sigma_t, \binom{k}{t})$. Thus in what follows below χ_t will be regarded as a map

$$\chi_t\colon\ \mathrm{Conf}(\mathbb{C}, k) \to \mathrm{Conf}\big(\mathbb{C}^t/\Sigma_t, \tbinom{k}{t}\big).$$

Addition of complex numbers provides a map

$$\oplus_t\colon\ \mathbb{C}^t/\Sigma_t \to \mathbb{C}$$

with

$$\oplus_t([z_1, \dots, z_t]) = z_1 + \cdots + z_t.$$

There is an induced map

$$\Theta_t\colon\ \mathrm{Conf}(\mathbb{C}, k) \to \mathbb{C}^{\binom{k}{t}}$$

given by the composite

$$\mathrm{Conf}(\mathbb{C}, k) \xrightarrow{\chi_t} (\mathbb{C}^t/\Sigma_t)^{\binom{k}{t}} \xrightarrow{(\oplus_t)^{\binom{k}{t}}} \mathbb{C}^{\binom{k}{t}}.$$

Thus

$$\Theta_t(z_1, z_2, \dots, z_k) = \prod_{i_1<i_2<\cdots<i_t} (z_{i_1} + z_{i_2} + \cdots + z_{i_t})$$

in $\mathbb{C}^{\binom{k}{t}}$.

The next proposition, a useful observation, is recorded next where

$$j\colon\ \mathrm{Conf}\big(\mathbb{C}, \tbinom{k}{t}\big) \to \mathbb{C}^{\binom{k}{t}}$$

is the natural inclusion. This observation is the starting point of the results here, and provides the basic motivation for considering the center of mass arrangement.

Proposition 1.1 *The following diagram is a pull-back (a cartesian diagram):*

$$\begin{tikzcd} M(t,k) \arrow[r] \arrow[d] & \mathrm{Conf}(\mathbb{C}, \binom{k}{t}) \arrow[d, "j"] \\ \mathrm{Conf}(\mathbb{C}, k) \arrow[r, "\Theta_t"] & \mathbb{C}^{\binom{k}{t}} \end{tikzcd}$$

Notice that $M(2,k)$ is the space of ordered k–tuples of distinct points such that no four of the points are the vertices of a possibly degenerate parallelogram. Consider the natural inclusion $M(2,k) \to \mathrm{Conf}(\mathbb{C},k)$ modulo the action of Σ_k the symmetric group on k letters

$$i(2,k)\colon\ M(2,k)/\Sigma_k \to \mathrm{Conf}(\mathbb{C},k)/\Sigma_k.$$

One question is whether there is a cross-section up to homotopy, or even a 2–local stable cross-section up to homotopy for this inclusion. This last question concerns plane geometry and whether the configuration space of distinct unordered k–tuples of points in the plane can be deformed to the subspace of points, no four of which are the vertices of a parallelogram.

Theorem 1.2 *If $k \geq 4$, the natural map*

$$i(2,k)\colon\ M(2,k)/\Sigma_k \to \mathrm{Conf}(\mathbb{C},k)/\Sigma_k$$

does not admit a surjection in mod–2 homology, and thus does not admit a cross-section (or a stable 2–local cross-section) up to homotopy.

The proof, homotopy theoretic without a specific computation of the homology of these spaces, gives features of the topology of double loop spaces which forces the maps $i(2,k)\colon\ M(2,k)/\Sigma_k \to \mathrm{Conf}(\mathbb{C},k)/\Sigma_k$ for $k \geq 4$ to fail to be epimorphisms in mod–2 homology. The analogous methods applied to the natural inclusion

$$i(p,k)\colon\ M(p,k)/\Sigma_k \to \mathrm{Conf}(\mathbb{C},k)/\Sigma_k$$

for p an odd prime fail to produce a non-trivial obstruction to the existence of a stable p–local section. Hence a problem unsolved here is whether $i(p,k)$ admits a stable p–local cross-section. The failure of the methods here in case p is an odd prime leads to the speculation in section 2 here concerning the localization of the double loop space of a sphere at an odd prime p.

Further properties of these arrangements are noted next. The natural "stabilization" map for configuration spaces fails to preserve the spaces $M(t,k)$. However, there are stabilization maps for the modified center of mass arrangements

$$S\colon\ M'(t,k) \to M'(t,k+1)$$

defined by

$$S(x_1,\ldots,x_k) = (x_1,\ldots,x_k,\vec{z})$$

where $\vec{z}$ is the vector $(L,0)$ with $L = 2t(1 + max_{k\geq i\geq 1}||x_i||)$. Notice that S takes values in $M'(t,k+1)$, but that the analogous map out of $M(t,k)$ takes values in $\mathrm{Conf}(\mathbb{C},k)$, but not in the subspace $M(t,k+1)$.

The next result follows directly from Cohen [2] and Cohen–May–Taylor [4].

Theorem 1.3 *The map*

$$S\colon\ M'(t,k) \to M'(t,k+1)$$

extends to a map

$$S_*\colon\ M'(t,k) \times_{\Sigma_k} Y^k \to\ M'(t,k+1) \times_{\Sigma_{k+1}} Y^{k+1}$$

which admits a stable left inverse for any path-connected CW-complex Y, and thus induces a split monomorphism in homology with any field coefficients.

Corollary 1.4 *The map $S\colon\ M'(t,k)/\Sigma_k \to\ M'(t,k+1)/\Sigma_{k+1}$ induces a split monomorphism in homology with coefficients in any graded permutation representation of Σ_k, and thus by specialization to either coefficients given by the trivial representation or the sign representation.*

Connections to homotopy theory as well as the motivation for considering the spaces $M(t,k)$ and the map

$$\chi_t\colon\ \mathrm{Conf}(\mathbb{C},k) \to (\mathbb{C}^t/\Sigma_t)^{\binom{k}{t}}$$

defined earlier in this section are given next. These connections arise from stable homotopy equivalences

$$H\colon\ \Omega^2\Sigma^2(X) \to \vee_{0\leq k} D_k(\Omega^2\Sigma^2(X))$$

in case X is a path-connected CW–complex originally proven by Snaith [10] and subsequently by Cohen, May and Taylor [4, 2] for which $D_k(\Omega^2\Sigma^2(X))$ is defined in Section 2 here.

This stable homotopy equivalence is obtained by adding maps given by

$$h_k\colon\ \Omega^2\Sigma^2(X) \to \Omega^{2k}\Sigma^{2k} D_k(\Omega^2\Sigma^2(X))$$

as observed in the appendix of [2]. These maps do not compress through

$$\Omega^{2k-1}\Sigma^{2k-1} D_k(\Omega^2\Sigma^2(X))$$

in case $k = 2^t$, and spaces are localized at the prime 2 (see Cohen and Mahowald [3]).

Specialize h_k to $k = p$ an odd prime and $X = S^{2n-1}$. The spaces $M(p,k)$ and $M'(p,k)$ as well as the map $\chi_t\colon\ \mathrm{Conf}(\mathbb{C},k) \to (\mathbb{C}^t/\Sigma_t)^{\binom{k}{t}}$ are introduced here in order to attempt to compress the maps

$$h_p\colon\ \Omega^2\Sigma^2(S^{2n-1}) \to \Omega^{2p}\Sigma^{2p}D_p(\Omega^2\Sigma^2(S^{2n-1}))$$

through some choice of map

$$\bar{h}_p\colon\ \Omega^2\Sigma^2(S^{2n-1}) \to \Omega^2\Sigma^2 D_p(\Omega^2\Sigma^2(S^{2n-1})).$$

The map h_p as given in [2, 4] is induced on the level of certain combinatorial models by the composite

$$\mathrm{Conf}(\mathbb{C},k) \xrightarrow{\chi_p} \mathrm{Conf}\big(\mathbb{C}^p/\Sigma_p, \tbinom{k}{p}\big) \xrightarrow{\text{inclusion}} (\mathbb{C}^p/\Sigma_p)^{\binom{k}{p}}.$$

A space $M_p(\mathbb{C},X)$ together with a map

$$I_p\colon\ M_p(\mathbb{C},X) \to \Omega^2\Sigma^2(X)$$

will be defined in Section 2 in which configuration spaces $\mathrm{Conf}(\mathbb{C},k)$ used in combinatorial models of $\Omega^2\Sigma^2(X)$ are replaced by the spaces $M(p,k)$. Furthermore, there are continuous maps

$$h_p\colon\ M_p(\mathbb{C},X) \to \Omega^2\Sigma^2(D_p(\Omega^2\Sigma^2(X))).$$

It is natural to compare the homotopy types of $\Omega^2 S^{2n+1}$ and $M_p(\mathbb{C},S^{2n-1})$ after localization at an odd prime p by the following theorem in which

$$E\colon\ \Sigma^2(Y) \to \Omega^{2p-2}\Sigma^{2p}(Y)$$

denotes the classical suspension map.

Theorem 1.5 *There is a commutative diagram*

$$\begin{tikzcd} M_p(\mathbb{C},X) \arrow[r, "h_p"] \arrow[d, "I_p"] & \Omega^2\Sigma^2(D_p(\Omega^2\Sigma^2(X))) \arrow[d, "\Omega^2(E)"] \\ C(\mathbb{C},X) \arrow[r, "h_p"] & \Omega^{2p}\Sigma^{2p}(D_p(\Omega^2\Sigma^2(X))). \end{tikzcd}$$

Thus if the map

$$I_p\colon\ M_p(\mathbb{C},S^{2n-1}) \to \Omega^2 S^{2n+1}$$

is a p–local equivalence, then there is a p–local map

$$\bar{h}_p\colon\ \Omega^2 S^{2n+1} \to \Omega^2\Sigma^2(D_p(\Omega^2\Sigma^2(S^{2n-1})))$$

which is a compression of the map $h_p\colon\ \Omega^2\Sigma^2(S^{2n-1}) \to \Omega^{2p}\Sigma^{2p}D_p(\Omega^2\Sigma^2(S^{2n-1}))$ and which induces an isomorphism on $H_{2np-2}(-;\mathbb{F}_p)$.

Some consequences of the existence of $\bar{h}_p$ are discussed in Section 2 here. These consequences suggest that it would be interesting to understand the behavior of the natural map

$$I_p\colon\ M_p(\mathbb{C}, S^{2n-1}) \to C(\mathbb{C}, S^{2n-1})$$

on the level of mod–p homology.

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2 Speculation concerning the localization of the double loop space of a sphere at an odd prime p, and applications

The main goal of this section is to point out that if the equivariant homology of either $M(t,k)$ or $M'(t,k)$ satisfies one statement below, then these spaces provide a method for constructing the localization at an odd prime p of the double loop space of an odd sphere. Some consequences are also given.

Let $R[\Sigma_k]$ denote the group ring of the symmetric group over a commutative ring R with 1, and let $\mathcal{S}$ denote a left $R[\Sigma_k]$–module. Let X denote a path-connected Hausdorff space with a free, right action of the symmetric group Σ_k. Let $H_*(X/\Sigma_k;\mathcal{S})$ denote the homology of the chain complex $C_*(X)\otimes_{\mathbb{Z}[\Sigma_k]}\mathcal{S}$ where $C_*(X)$ denotes the singular chain complex of X.

Observe that the natural inclusion

$$M(t,k) \to \mathrm{Conf}(\mathbb{C},k)$$

induces a homomorphism

$$H_*(M(t,k)/\Sigma_k;\mathcal{S}) \to\ H_*(\mathrm{Conf}(\mathbb{C},k)/\Sigma_k;\mathcal{S}).$$

If t is equal to an odd prime p, and $\mathcal{S}$ is the coefficient module given by $\mathbb{F}_p(\pm 1)$ the field of p–elements with the action of Σ_k specified by the sign representation, then one question is to decide whether this map induces an isomorphism in mod–p homology.

There is no strong evidence either way, although an affirmative answer has interesting consequences which are described below. The analogous question for $p = 2$ fails at once by Theorem 1.2.

The reason for the interest in these particular homology groups is the following observation implicit in Cohen [1] as follows.

Theorem 2.1 *Let $\mathbb{F}$ denote a field. For each integer i greater than 0, there is an isomorphism*

$$\oplus_{k\geq 0}H_{i-k(2n-1)}(\mathrm{Conf}(\mathbb{C},k)/\Sigma_k,\mathbb{F}(\pm 1)) \to\ H_i(\Omega^2 S^{2n+1};\mathbb{F}).$$

The next corollary follows at once.

Corollary 2.2 *Let $\mathbb{F}$ denote a field, and p an odd prime.*

(1) *If the natural inclusion*

$$M(p,k)\to \mathrm{Conf}(\mathbb{C},k)$$

induces an isomorphism

$$H_*(M(p,k)/\Sigma_k;\mathbb{F}_p(\pm 1))\to H_*(\mathrm{Conf}(\mathbb{C},k)/\Sigma_k;\mathbb{F}_p(\pm 1)),$$

then there are isomorphisms

$$\oplus_{k\geq 0}H_{i-k(2n-1)}(M(p,k)/\Sigma_k,\mathbb{F}_p(\pm 1))\to H_i(\Omega^2 S^{2n+1};\mathbb{F}_p).$$

(2) *If the natural inclusion*

$$M'(p,k)\to \mathrm{Conf}(\mathbb{R}^2,k)$$

induces an isomorphism

$$H_*(M'(p,k)/\Sigma_k;\mathbb{F}_p(\pm 1))\to H_*(\mathrm{Conf}(\mathbb{C},k)/\Sigma_k;\mathbb{F}_p(\pm 1)),$$

then there are isomorphisms

$$\oplus_{k\geq 0}H_{i-k(2n-1)}(M'(p,k)/\Sigma_k,\mathbb{F}_p(\pm 1))\to H_i(\Omega^2 S^{2n+1};\mathbb{F}_p).$$

The spaces $M(p,k)$ and $M'(p,k)$ are used next to give analogues of labeled configuration spaces in which the configuration space itself is replaced by a "center of mass construction" as given above. Let Y denote a pointed space with base-point $*$ and W any topological space. Recall the labeled configuration space

$$C(W,Y)$$

given by equivalence classes of pairs $[S,f]$ where

(1) S is a finite subset of W,

(2) $f\colon S \to Y$ is a function, and

(3) $[S, f]$ is equivalent to $[S - \{p\}, f|_{S-\{p\}}]$ if and only if $f(p) = *$.

One theorem proven by May [7] is as follows.

Theorem 2.3 *If Y is a path-connected CW-complex, then $C(\mathbb{R}^n, Y)$ is homotopy equivalent to $\Omega^n\Sigma^n(Y)$*

Technically, May's proof does not exhibit a map between these two spaces. There are weak equivalences on the level of May's construction [7] $\alpha\colon C_n(Y) \to \Omega^n\Sigma^n(Y)$ and the natural evaluation map $e\colon C_n(Y) \to C(\mathbb{R}^n, Y)$.

Furthermore, the construction $D_k(\Omega^2\Sigma^2(X))$ is homotopy equivalent to

$$\mathrm{Conf}(\mathbb{C}, k) \times_{\Sigma_k} X^{(k)} / \mathrm{Conf}(\mathbb{C}, k) \times_{\Sigma_k} \{*\}$$

for which $X^{(k)}$ denotes the k–fold smash product [7]. When localized at an odd prime p, $D_p(\Omega^2 S^{2n+1})$ is homotopy equivalent to a mod–p Moore space $P^{2np-1}(p)$ with a single non-vanishing reduced homology group given by $\mathbb{Z}/p\mathbb{Z}$ in dimension $2np-2$. This last assertion follows from the computations in Cohen [1].

Definition 2.4 Define

$$M_t(\mathbb{C}, Y)$$

to be the subspace of $C(\mathbb{C}, Y)$ given by those points for which S is a subset of $M(t, k)$ with natural inclusion denoted by $I_p\colon M_t(\mathbb{C}, Y) \to C(\mathbb{C}, Y)$ and

$$M'_t(\mathbb{C}, Y)$$

to be the subspace of $C(\mathbb{C}, Y)$ given by those points for which S is a subset of $M'(t, k)$ with natural inclusion denoted (ambiguously) by $I_p\colon M_t(\mathbb{C}, Y) \to C(\mathbb{C}, Y)$.

The next statement provides a potential method for constructing the localization at p of the double loop space of an odd sphere which also has some useful properties.

Theorem 2.5 *Assume that p is an odd prime.*

(1) *If* $M(t,k) \to \mathrm{Conf}(\mathbb{C},k)$ *induces an isomorphism*

$$H_*(M(t,k)/\Sigma_k;\mathbb{F}_p(\pm 1)) \to H_*(\mathrm{Conf}(\mathbb{C},k)/\Sigma_k;\mathbb{F}_p(\pm 1))$$

for t *an odd prime* p, *then the natural map*

$$I_p\colon\ M_p(\mathbb{C},S^{2n-1}) \to\ \Omega^2 S^{2n+1}$$

induces a mod–p *homology isomorphism.*

(2) *If* $M'(t,k) \to \mathrm{Conf}(\mathbb{C},k)$ *induces an isomorphism*

$$H_*(M'(t,k)/\Sigma_k;\mathbb{F}_p(\pm 1)) \to H_*(\mathrm{Conf}(\mathbb{C},k)/\Sigma_k;\mathbb{F}_p(\pm 1))$$

for t *an odd prime* p, *then the natural map*

$$I_p\colon\ M'_p(\mathbb{C},S^{2n-1}) \to\ \Omega^2 S^{2n+1}$$

induces a mod–p *homology isomorphism.*

One consequence of this last theorem is that it implies properties of the double suspension of $E^2\colon\ S^{2n-1} \to \Omega^2 S^{2n+1}$ after localization at an odd prime. In particular, the next corollary follows directly.

Corollary 2.6 *Let* p *denote an odd prime. If either*

(1) *the natural inclusion* $M(p,k) \to \mathrm{Conf}(\mathbb{C},k)$ *induces an isomorphism*

$$H_*(M(p,k)/\Sigma_k;\mathbb{F}_p(\pm 1)) \to H_*(\mathrm{Conf}(\mathbb{C},k)/\Sigma_k;\mathbb{F}_p(\pm 1)),$$

or

(2) *the natural inclusion* $M'(p,k) \to \mathrm{Conf}(\mathbb{R}^2,k)$ *induces an isomorphism*

$$H_*(M'(p,k)/\Sigma_k;\mathbb{F}_p(\pm 1)) \to H_*(\mathrm{Conf}(\mathbb{C},k)/\Sigma_k;\mathbb{F}_p(\pm 1)),$$

then after localization at p, *the mod–*p *Moore space*

$$P^{2np+1}(p)$$

is a retract of $\Sigma^2\Omega^2 S^{2n+1}$. *In that case, the following hold:*

(1) *Any map*

$$\alpha\colon\ P^{2p+1}(p) \to S^3$$

given by an extension of $\alpha_1\colon\ S^{2p} \to S^3$, *which realizes the first element of order* p *in the homotopy groups of the* 3*–sphere induces a split epimorphism on the* p*–primary component of homotopy groups.*

(2) *After localization at the prime p, the homotopy theoretic fibre of the double suspension $E^2\colon\ S^{2n-1} \to \Omega^2 S^{2n+1}$ is the fibre of a map $\Omega^2 S^{2np+1} \to S^{2np-1}$.*

Remarks 2.7

(1) The main content of Corollary 2.6 is that the “center of mass arrangements" may provide a useful way to construct a localization at odd primes for the double loop space of an odd sphere. Corollary 2.6 follows from Theorem 1.5 as outlined in Cohen [2].

(2) In addition, Theorem 1.2 shows that these constructions fail to give the localization at the prime 2 of $\Omega^2 S^{2n+1}$.

(3) It would also be interesting to see whether there are analogous properties for the center of mass arrangement with $\mathbb{C}$ replaced by $\mathbb{C}^n$ which may provide the localization of $\Omega^{2n} S^{2(n+k)+1}$ at an odd prime p.

3 Sketch of Proposition 1.1

Notice that the set theoretic pull-back in the diagram given in Proposition 1.1 is precisely the subspace of the configuration space given by the t–fold center of mass arrangement $M(t,k)$.

4 Calculations at the prime 2, and the proof of Theorem 1.2

The method here of comparing the homology of the center of mass arrangement with that of the configuration space uses some additional topology. Here, consider the natural inclusion $M(p,k) \to \mathrm{Conf}(\mathbb{C},k)$ together with the induced map

$$I_p\colon\ M_p(\mathbb{C}, S^{2n-1}) \to C(\mathbb{C}, S^{2n-1}).$$

The space $C(\mathbb{C}, S^{2n-1})$ is homotopy equivalent to $\Omega^2 S^{2n+1}$ (see May [7]). In addition, the spaces $M_p(\mathbb{C}, S^{2n-1})$, and $C(\mathbb{C}, S^{2n-1})$ admit stable decompositions which are compatible by the remarks in Cohen [2] and Cohen–May–Taylor [4]. Notice that the inclusion $M(t,k) \to \mathrm{Conf}(\mathbb{C},k)$ is the identity in case $k \le t$ by definition of $M(t,k)$. Thus the induced maps on stable summands

$$D_j(M_p(\mathbb{C}, S^{2n-1})) \to D_j(C(\mathbb{C}, S^{2n-1}))$$

is the identity in case $j \leq p$, a feature which is used below.

Let $p = 2$, and consider the second stable summand

$$D_2(M_2(\mathbb{C}, S^{2n-1})) = D_2(C(\mathbb{C}, S^{2n-1})).$$

This stable summand is homotopy equivalent to

$$(S^1 \times_{\Sigma_2} S^{4n-2})/(S^1 \times_{\Sigma_2} *)$$

which is itself homotopy equivalent to

$$\Sigma^{4n-3}(\mathbb{RP}^2).$$

Let u denote a basis element for $H_{4n-2}(\Sigma^{4n-3}(\mathbb{RP}^2); \mathbb{F}_2)$, and v denote a basis element for $H_{4n-1}(\Sigma^{4n-3}(\mathbb{RP}^2); \mathbb{F}_2)$.

In addition, there is a strictly commutative diagram

$$\begin{tikzcd}
M_2(\mathbb{C}, S^{2n-1}) \arrow[r, "h_2"] \arrow[d, "I_2"] & \Omega^\infty\Sigma^\infty(D_2(C(\mathbb{C}, S^{2n-1}))) \arrow[d, "1"] \\
C(\mathbb{R}^2, S^{2n-1}) \arrow[r, "h_2"] & \Omega^\infty\Sigma^\infty(D_2(C(\mathbb{C}, S^{2n-1})))
\end{tikzcd}$$

which gives the fact that the space $D_2(C(\mathbb{C}, S^{2n-1})) = \Sigma^{4n-3}(\mathbb{RP}^2)$ is a stable retract of both spaces, in a way which is compatible with the natural stable decompositions.

Further, by Proposition 1.1, together with the definition [2] of the map

$$h_2\colon\ M_2(\mathbb{C}, S^{2n-1}) \to \Omega^\infty\Sigma^\infty(D_2(C(\mathbb{C}, S^{2n-1}))),$$

there is a compression of this map through $\Omega^2\Sigma^2(D_2(C(\mathbb{C}, S^{2n-1})))$. Thus, there is a commutative diagram given as follows.

$$\begin{tikzcd}
M_2(\mathbb{C}, S^{2n-1}) \arrow[r, "h_2"] \arrow[d, "I_2"] & \Omega^2\Sigma^2(D_2(C(\mathbb{C}, S^{2n-1}))) \arrow[d, "\Omega^2(E)"] \\
C(\mathbb{R}^2, S^{2n-1}) \arrow[r, "h_2"] & \Omega^\infty\Sigma^\infty(D_2(C(\mathbb{C}, S^{2n-1}))).
\end{tikzcd}$$

These remarks have the following consequence for which $Q_i(x)$ is the standard notation for Araki–Kudo–Dyer–Lashof operations as described in [1].

Lemma 4.1 *The image of the map*

$$h_2\colon\ M_2(\mathbb{C}, S^{2n-1}) \to \Omega^\infty\Sigma^\infty(D_2(M_2(\mathbb{C}, S^{2n-1})))$$

on the level of mod–2 homology is contained in the subalgebra generated by the elements x, and $Q_1^q(x)$ for $q \geq 1$ for which x is an element of a basis for the mod–2 homology of $\Sigma^{4n-3}(\mathbb{RP}^2)$ given by $\{u, v\}$. In particular, the element $Q_3(x)$ cannot appear as a non-trivial summand of the image.

Lemma 4.2 *If $k \geq 4$, and the natural map*

$$M(2,k)/\Sigma_k \to \mathrm{Conf}(\mathbb{C},k)/\Sigma_k$$

induces a surjection in mod–2 homology, then the natural map

$$M(2,4)/\Sigma_4 \to \mathrm{Conf}(\mathbb{C},4)/\Sigma_4$$

induces a surjection in mod–2 homology.

Proof Notice that if $k \geq 4$, the space $\mathrm{Conf}(\mathbb{C},4)/\Sigma_4$ is a stable retract of the space $\mathrm{Conf}(\mathbb{C},k)/\Sigma_k$ via a map induced by the transfer obtained from the natural Σ_k–cover (see Cohen–May–Taylor [5]). Thus there is a commutative diagram

$$\begin{array}{ccc}
\Sigma^{2k}(M(2,k)/\Sigma_k) & \longrightarrow & \Sigma^{2k}(\mathrm{Conf}(\mathbb{C},k)/\Sigma_k) \\
\big\downarrow{\scriptstyle tr} & & \big\downarrow{\scriptstyle tr} \\
\Sigma^{2k}(M(2,4)/\Sigma_4) & \longrightarrow & \Sigma^{2k}(\mathrm{Conf}(\mathbb{C},4)/\Sigma_4)
\end{array}$$

in which the vertical maps are induced by the natural transfer. Hence the natural map $M(2,4)/\Sigma_4 \to \mathrm{Conf}(\mathbb{C},4)/\Sigma_4$ induces a surjection on mod–2 homology as the maps $M(2,k)/\Sigma_k \to \mathrm{Conf}(\mathbb{C},k)/\Sigma_k$ as well as $tr\colon \Sigma^{2k}(\mathrm{Conf}(\mathbb{C},k)/\Sigma_k) \to \Sigma^{2k}(\mathrm{Conf}(\mathbb{C},4)/\Sigma_4)$ induce surjections on mod–2 homology by [5]. $\square$

The proof of Theorem 1.2 is given next.

Proof Assume that the natural inclusion $M(2,k) \to \mathrm{Conf}(\mathbb{C},k)$ induces an epimorphism on the level of $H_*(M(2,k)/\Sigma_k;\mathbb{F}_2) \to H_*(\mathrm{Conf}(\mathbb{C},k)/\Sigma_k;\mathbb{F}_2)$ for some $k \geq 4$. Then by Lemma 4.2, the natural map $M(2,4)/\Sigma_4 \to \mathrm{Conf}(\mathbb{C},4)/\Sigma_4$ induces a surjection on mod–2 homology, and the induced map $H_*(M_2(\mathbb{C},S^{2n-1});\mathbb{F}_2) \to H_*(C(\mathbb{C},S^{2n-1});\mathbb{F}_2)$ is an epimorphism in dimensions $\leq 8n-1$. This will lead to a contradiction.

Notice that

(1) $h_{2*}({x_{2n-1}}^2) = u$,

(2) $h_{2*}(Q_1(x_{2n-1})) = v$ and

(3) $h_{2*}(Q_1Q_1(x_{2n-1})) = AQ_1(v)+BQ_3(u)$ for scalars A, and B where u is the unique non-zero class in $H_{4n-2}(D_2(C(\mathbb{R}^2, S^{2n-1})); \mathbb{F}_2)$, and v is the unique non-zero class in $H_{4n-1}(D_2(C(\mathbb{C}, S^{2n-1})); \mathbb{F}_2)$ (see Cohen [1]).

A direct computation using Sq^1_*, Sq^2_*, and the coproduct gives

$$A = B = 1.$$

The details are as follows. Notice that $Sq^2_*(Q_1Q_1(x_{2n-1})) = 0$, but that

$$Sq^2_*(Q_1(v)) = Q_1(u) = Sq^2_*(Q_3(u)).$$

Thus $A = B$. Furthermore $Sq^1_*(Q_1Q_1(x_{2n-1})) = Q_1(x_{2n-1})^2$.

Finally notice that $h_{2*}((x_{2n-1}{}^2) \cdot Q_1(x_{2n-1})) = u \cdot v + P$ where P is a primitive element. The only non-zero choice for this primitive element P is $Q_1(u)$. However, $Sq^1_*(P) = 0$. Thus $h_{2*}(x_{2n-1}{}^4) = Sq^1_*(u \cdot v + P) = u^2$. Hence

$$Sq^2_*Sq^1_*h_{2*}(Q_1Q_1(x_{2n-1})) = u^2$$

and $A = B = 1$.

It follows that if the natural map $M(2,4)/\Sigma_4 \to \mathrm{Conf}(\mathbb{C},4)/\Sigma_4$ induces a surjection on mod–2 homology, then the class $Q_1(v) + Q_3(u)$ is in the image of the composite of the following two maps:

$$I_{2*}\colon\ H_*(M_2(\mathbb{C}, S^{2n-1}); \mathbb{F}_2) \to H_*(C(\mathbb{C}, S^{2n-1}); \mathbb{F}_2),$$

and

$$h_{2*}\colon\ H_*(C(\mathbb{C}, S^{2n-1}); \mathbb{F}_2) \to H_*(\Omega^\infty\Sigma^\infty D_2(C(\mathbb{C}, S^{2n-1})); \mathbb{F}_2).$$

Thus the above computation gives that the class $Q_1(v) + Q_3(u)$ is in the image of the composite

$$H_*(M_2(\mathbb{C}, S^{2n-1}); \mathbb{F}_2) \xrightarrow{h_{2*}\circ I_{2*}} H_*(\Omega^\infty\Sigma^\infty D_2(C(\mathbb{C}, S^{2n-1})); \mathbb{F}_2).$$

By Lemma 4.1, the class $Q_1(v) + Q_3(u)$ cannot be in the image of the map

$$H_*(\Omega^2\Sigma^2(D_2(M_2(\mathbb{C}, S^{2n-1}))); \mathbb{F}_2) \to H_*(\Omega^\infty\Sigma^\infty(D_2(C(\mathbb{C}, S^{2n-1}))); \mathbb{F}_2).$$

Hence, (1) the natural map $M(2,4)/\Sigma_4 \to \mathrm{Conf}(\mathbb{C},4)/\Sigma_4$ cannot induce a surjection on mod–2 homology and (2) the natural map $M(2,k)/\Sigma_k \to \mathrm{Conf}(\mathbb{C},k)/\Sigma_k$ for $k \geq 4$ cannot induce a surjection on mod–2 homology. The theorem follows. □

5 Sketch of Theorem 1.3 and Corollary 1.4

The proof of follows Theorem 1.3 at once from the constructions in the appendix of Cohen [2] or the main theorem in Cohen–May–Taylor [4] where it was shown that these maps admit stable right homotopy inverses.

To check Corollary 1.4, notice that the sign representation is given by the action of the symmetric group on the top non-vanishing homology group of $(S^1)^n$. The corollary follows from Theorem 1.3.

6 Sketch of Theorem 1.5

The commutativity of the diagram in Theorem 1.5 follows by definition. That the map

$$h_p\colon\ \Omega^2 S^{2n+1} \to \Omega^{2p}\Sigma^{2p}(D_p(\Omega^2\Sigma^2(S^{2n-1})))$$

induces an isomorphism on $H_{2np-2}(-;\mathbb{F}_p)$ is checked in Cohen [2]. Since h_p induces an isomorphism on $H_{2np-2}(-;\mathbb{F}_p)$, it follows from the known homology of these spaces that $\bar{h}_p$ does also. Given a map with the homological properties of $\bar{h}_p$, the proof of Theorem 1.5 follows from [2].

Remark 6.1 The goal of this approach is to try to desuspend a map analogous to that given by Selick [9].

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